From symplectic deformation to isotopy

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Abstract

Let X be an oriented 4-manifold which does not have simple SW-type, for example a blow-up of a rational or ruled surface. We show that any two cohomologous and deformation equivalent symplectic forms on X are isotopic. This implies that blow-ups of these manifolds are unique, thus extending work of Biran. We also establish uniqueness of structure for certain fibered 4-manifolds.

1 Introduction

Two symplectic forms ω_0, ω_1 on X are said to be **deformation equivalent** if they may be joined by a family of symplectic forms, and are called **isotopic** if this family may be chosen so that its elements all lie in the same cohomology class. Moser showed that every isotopy on a compact manifold has the form $\phi_t^*(\omega_0)$, where $\phi_t : X \to X$ is a family of diffeomorphisms starting at the identity. Examples are known in dimensions 6 and above of cohomologous symplectic forms that are deformation equivalent but not isotopic: see [16, Example 7.23]. No such examples are known in dimension 4, and it is a possibility that the two notions are the same in this case. (Note that the Gromov invariants are deformation invariants – in fact, Taubes's recent work shows that they are smooth invariants – and so they cannot help with this question.)

In [6] Lalonde–McDuff developed ideas from [5, 10] into an "inflation" procedure that converts a deformation into an isotopy, and applied it to establish the uniqueness of symplectic structures on ruled surfaces. The present note extends the range of this procedure and describes various applications of it.

Here is the basic lemma: its proof is sketched in §2 below. We will denote the Poincaré dual of a class A by $\mathrm{PD}(A)$, and will express the result in terms of

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the Gromov invariant $\mathrm{Gr}_0(A)$ that counts *connected* J-holomorphic representatives of the class A. (This is the invariant used by Ruan in [19]. To a first approximation, it is the same as the invariant used by Taubes.)

Lemma 1.1 (Inflation Lemma) *Let A be a class in $H_2(X, \mathbf{Z})$, with nonnegative self-intersection number and nonzero Gromov invariant $\mathrm{Gr}_0(A)$. Moreover, if $A^2 = 0$ assume that A is a primitive class. Then, given any family $\omega_t, 0 \le t \le 1$, of symplectic forms on X with $\omega_0 = \omega$, there is a family ρ_t of closed forms on X in class $\mathrm{PD}(A)$ such that the family*

$$\omega_t + \kappa(t)\rho_t, \quad 0 \le t \le 1,$$

is symplectic whenever $\kappa(t) \ge 0$.

To apply this lemma, we need to understand the Gromov invariants of X. Recall that a symplectic 4-manifold X is said to have **simple SW-type** or just **simple type** if its only nonzero Gromov invariants occur in classes $A \in H_2(X)$ for which

$$k(A) = -K \cdot A + A^2 = 0.$$

Taubes showed in [20, 21] that all symplectic 4-manifolds with $b_2^+ > 1$ have simple type. It follows easily from the wall crossing formula of Li–Liu [7, 8] that if $b_2^+ = 1$ then X has simple type only if $b_1 \ne 0$ and all products of elements in $H^1(X)$ vanish. Moreover, Liu showed in [9] that any minimal symplectic manifold with $K^2 < 0$ is ruled. Therefore the symplectic 4-manifolds with nonsimple type are blow-ups of

(i) rational and ruled manifolds;

(ii) manifolds (such as the Barlow or Enriques surface) with $b_1 = 0, b_2^+ = 1$; and

(iii) manifolds with $b_1 = 2$ and $(H^1(X))^2 \ne 0$. Examples with $K = 0$ are hyperelliptic surfaces and some (but not all) of the non-Kähler T^2-bundles over T^2. There are also Kähler examples with $K \ne 0$, for example quotients of the form $T^2 \times \Sigma/G$ where Σ has genus > 1 and G is a suitable finite group.[1]

We will show below that manifolds which do not have simple type have enough nonzero Gromov invariants for the following result to hold.

Theorem 1.2 *Let (X, ω) be a symplectic 4-manifold which does not have simple type. Then any deformation between two cohomologous symplectic forms on X may be homotoped through deformations with fixed endpoints to an isotopy.*

Corollary 1.3 *For any $k > 0$ and any X not of simple type, there is at most one way of blowing up k points to specified sizes.*

[1] Le Hong Van pointed out that such examples exist: they were omitted from the survey article [17].

Proof: It is easy to see that any two blow-ups of a symplectic manifold are deformation equivalent. (For more details of this step see the proof of Corollary 1.5 below.) The cohomology class of the blow-up is determined by the "size" of the blown-up points. Hence the above theorem implies that cohomologous blow-ups are isotopic. □

When $k = 1$ this corollary was first proved by McDuff [11, 16] in the case of $\mathbf{C}P^2$ and ruled surfaces over S^2, and by Lalonde [5] for (certain) irrational ruled surfaces. The methods used were very geometric. Recently, Biran [1] established uniqueness for many blow-ups of $\mathbf{C}P^2$. Independently, Gatien [3] realised that arguments similar to the ones in this note should lead to the uniqueness of blow-ups for ruled manifolds.

The next result was proved by Lalonde–McDuff in [6].

Corollary 1.4 *Let X be oriented diffeomorphic to a minimal rational or ruled surface, and let $a \in H^2(X)$. Then there is a symplectic form on X in the class a which is compatible with the given orientation if and only if $a^2 > 0$. Moreover, any two symplectic forms in class a are diffeomorphic.*

Proof: The first result is obvious, and the second holds because, by [10], given any two symplectic forms ω_0, ω_1 on X there is a diffeomorphism ϕ of X such that $\phi^*(\omega_0)$ is deformation equivalent to ω_1. □

Embedding balls and blowing up

Using the correspondence between embedding balls and blowing up that was first described in [11], we can deduce the connectedness of the space

$$Emb(\coprod_{i=1}^{k} B(\lambda_i), X)$$

of symplectic embeddings of the disjoint union of the closed 4-balls $B(\lambda_i), i = 1, \dots, k$, of radius λ_i into X, with the C^1-topology.

Corollary 1.5 *If X has nonsimple type, $Emb(\coprod_{i=1}^{k} B(\lambda_i), X)$ is path-connected.*

The proof is sketched later. Note that a similar result holds for manifolds of the form $X - Z$ where X has nonsimple type as above and Z is any symplectic submanifold. In particular, it holds for the open unit ball, which is just $\mathbf{C}P^2 - \mathbf{C}P^1$.

Remark 1.6 Here we have applied the inflation procedure to discuss the uniqueness problem for blow-ups. Closely related is the packing problem: what constraints are there on the radii of balls that embed symplectically and disjointly in X? The paper [15] gives a complete solution of this problem for $k \leq 9$ balls

in $\mathbf{C}P^2$, but left open the question of whether there is a full filling (i.e. a filling that uses up all the volume) of $\mathbf{C}P^2$ by k equal balls for $k \geq 10$. Using the methods of the present paper, Biran has shown that such a filling does indeed exist: see [2].

The symplectic cone

Our methods also give some information for general symplectic 4-manifolds. As Biran points out in his discussion of the packing problem, one can use Lemma 1.1 to get information on the symplectic cone

$$\mathcal{C}_X = \{a \in H^2(X;\mathbf{R}) : a \text{ has a symplectic representative } \}.$$

(Here we consider only those symplectic forms that are compatible with the given orientation of X.) Because, by Taubes, the set of classes with nontrivial Gromov invariants is an invariant of the smooth structure of X, it is clear that

$$\mathcal{C}_X \subset \{a : a(A) > 0 \text{ for all } A \neq 0 \text{ such that } \mathrm{Gr}\,(A) \neq 0\}.$$

Proposition 1.7 *(i) If there is some symplectic form ω on X for which the class A is represented by a symplectic submanifold with all components of non-negative self-intersection, then* $\mathrm{PD}(A)$ *is in the closure* $\overline{\mathcal{C}}_X$ *of the symplectic cone* $\mathcal{C}_X$. *Moreover, if X has simple type and* $\mathrm{Gr}\,(A) \neq 0$ *for some class $A \neq K, 0$, then* $\mathrm{PD}(A) \notin \mathcal{C}_X$.

(ii) Suppose that X is minimal with $b_2^+ > 1$ and that K is nontorsion. Then $\mathrm{PD}(K) \in \overline{\mathcal{C}}_X$, *while $K \notin \mathcal{C}_X$ if there is any nonzero class A with $A^2 = 0$ and* $\mathrm{Gr}\,(A) \neq 0$.

Proof: The first statement in (i) follows immediately from the Inflation Lemma, because the class $\frac{1}{\kappa}[\omega + \kappa\rho]$ converges to $[\rho]$ as $\kappa \to \infty$.[2] If $\mathrm{Gr}\,(A) \neq 0$ then the fact that X has simple type immediately implies that $A \cdot (K - A) = 0$. Moreover, Taubes showed that $\mathrm{Gr}\,(A) = \pm\mathrm{Gr}\,(K - A)$. Therefore, if both A and $K - A$ are nonzero, $\mathrm{PD}(A) \notin \mathcal{C}_X$. (ii) follows immediately from Taubes' results since the hypotheses imply that $K \cdot A = 0$. □

It would be interesting to know whether $K \in \mathcal{C}_X$ in the case when K has a representative with all components of positive self- intersection.

[2] We do not need any hypothesis here about nonmultiply covered toral components since we are applying this lemma using a single submanifold Z rather than a family Z_t.

Deformations on arbitrary X

Let us write $\mathcal{E}$ for the subset of $H_2(X, \mathbf{Z})$ consisting of all classes that can be represented by a symplectically embedded 2-spheres of self-intersection -1. If X is not the blow-up of a rational or ruled manifold, it was shown in McDuff [12] that $\mathcal{E}$ is finite and consists of mutually orthogonal classes, ie $E \cdot E' = 0$ for $E \neq E' \in \mathcal{E}$. In particular, X is the symplectic blow-up of a unique minimal manifold Y which is obtained by blowing down a set of representatives of the classes in $\mathcal{E}$. Now, the Seiberg–Witten blow-up formulas imply that the Gromov invariants of X are determined by those of Y by the following rule:[3] there is a natural decomposition

$$H_2(X; \mathbf{Z}) = H_2(Y; \mathbf{Z}) \oplus \sum_{E \in \mathcal{E}} \mathbf{Z}E,$$

and for each class $B \in H_2(Y; \mathbf{Z})$

$$\begin{aligned} \mathrm{Gr}_X(B + \sum_{E \in \mathcal{E}} \varepsilon_E E) &= \mathrm{Gr}_Y(B) \quad \text{if } \varepsilon_E = 0 \text{ or } 1 \quad \text{for all } E, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Taubes showed in [21] that each J-holomorphic representative of the class B consists of a finite number of disjoint components with nonnegative self-intersection, and that the class $B + \sum \varepsilon_E E$ is represented by adjoining the appropriate exceptional curves. Moreover, these components are embedded, except possibly if they represent non-primitive classes T with $T^2 = 0$, in which case they may be multiply-covered tori of self-intersection zero. Since these multiply-covered tori do not always persist under deformation (see [24]), we cannot use them. Therefore, as before, we use the invariant $\mathrm{Gr}_0(A)$, which is defined to be the algebraic number of *connected* J-holomorphic representatives of the class A if either $A^2 > 0$ or A is primitive. We then define

$$V = V(X) \subset H^2(X, \mathbf{R})$$

to be the convex cone spanned by the Poincaré duals of those classes A such that

(i) $A \cdot E \geq 0$ for all $E \in \mathcal{E}$;
(ii) either A is primitive or $A^2 > 0$; and
(iii) $\mathrm{Gr}_0(A) \neq 0$.

Remark 1.8 It is not hard to see (and details may be found in [14]) that if A has no J-holomorphic representatives that include components that are

[3] Note that the theory of J-holomorphic curves implies only that $\mathcal{E}$ and hence Y depend on the deformation class of ω, while Taubes–Seiberg–Witten theory shows that when $b_2^+ > 1$ they are smooth invariants of X.

multiply-covered tori then

$$\mathrm{Gr}\,(A) = \sum_{A_1+\ldots+A_k=A} \prod_i \mathrm{Gr}\,_0(A_i),$$

where the sum is taken over all sets $\{A_1, \ldots, A_k\}$ such that $A = A_1 + \ldots A_k$. Further, if all regular multiply-covered J-holomorphic tori in X lie in classes which are multiples of primitive classes with nonzero Gromov invariants, then V may be defined by replacing the conditions (ii) and (iii) above by

(iv) $\mathrm{Gr}\,(A) \neq 0$.

Given these definitions, the inflation procedure applies immediately to show:

Theorem 1.9 *Let X be any symplectic 4-manifold. If $\omega_t, t \in [0,1]$, is a deformation with $[\omega_0] = [\omega_1]$ such that*

$$[\omega_0] \in \mathbf{R}^+[\omega_t] + V$$

for all t, then ω_t can be homotoped (rel endpoints) to an isotopy.

The above theorem does not give any interesting information about blow-ups. For V lies in the annihilator of $\mathcal{E}$ so that we cannot now change the size of the exceptional curves. However, as we now show, one can sometimes combine this result with other geometric information about the symplectic structure to get something new.

Symplectic fibrations

Let $\pi : X \to B$ be a fibration with compact oriented total space X and oriented base B. A symplectic form ω on X is said to be π-compatible if all the fibers of π are symplectic submanifolds of X and if the orientations that ω defines on X and the fibers equal the given ones. We will see in §3 that when the base and fiber have dimension 2 all such forms are deformation equivalent. Our methods allow us to change this deformation into an isotopy when X has nonsimple type (for example, if the base or fiber is a sphere or if X is a hyperelliptic surface) and also in some other cases. Here is a sample result that applies when $X = F \times B$. We write σ_F, σ_B for the pullback to X of area forms on F and B that have total area 1 and g_F, g_B for the genus of F and B.

Proposition 1.10 *Suppose that ω is a π-compatible form on the product $X = F \times B$ where $g_F > 1$. If $[\omega] = [\sigma_F + \lambda\sigma_B]$, where $\lambda(g_F - 1) > g_B - 1$, then ω is isotopic to the split form $\sigma_F + \lambda\sigma_B$.*

The proof is given in §3.

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2 Inflation and manifolds of nonsimple type

We begin by sketching the proof of the inflation lemma.

Proof of Lemma 1.1

Let ω_t be any deformation with $[\omega_0] = [\omega_1]$, and choose a generic family J_t of ω_t-tame almost complex structures. Lalonde and McDuff show in [6] that if A with $\mathrm{Gr}\,(A) \neq 0$ may be represented by a symplectically embedded submanifold of positive self-intersection then, for some smooth map $t \mapsto \mu(t)$ with $\mu(0) = 0, \mu(1) = 1$, there is a smooth family Z_t of $J_{\mu(t)}$-holomorphic embedded submanifolds of X in class A. The same argument holds when $A^2 = 0$ provided that A is primitive. [4] Without loss of generality we may suppress the reparametrization, and suppose that $\mu(t) = t$. It is shown in Lemma 3.7 of [13] that there is a smooth family ρ_t of closed 2-forms on X that represent the class $\mathrm{PD}(A)$ and are such that

$$\omega_t + \kappa(t)\rho_t$$

is symplectic for all constants $\kappa(t) \geq 0$ and all $t \in [0, 1]$. For completeness, we sketch this construction for fixed t, noting that the construction can be made to vary smoothly with t.

The form ρ is supported near Z and represents the Thom class of the normal bundle to Z. If this normal bundle is trivial, a neighborhood of Z is symplectically equivalent to a product $Z \times D^2$ and the existence of ρ is obvious. In general, let $k = Z \cdot Z$ and choose a connection γ on the normal circle bundle $\pi : Y \to Z$ such that $d\gamma = -\pi^*(f\omega_Z)$, where $\omega_Z = \omega|_Z$ and the function $f \geq 0$ has appropriate integral over Z. Then a neighborhood of Z can be symplectically identified with a neighborhood of the zero section in the associated complex line bundle, equipped with the symplectic form

$$\tau = \pi^*(\omega_Z) + d(\pi r^2 \gamma),$$

where r is the radial distance function. Hence one can take ρ to be given by the formula $-d(g(r)\gamma)$, where $g(r)$ is a nonnegative function with support in a small interval $[0, \varepsilon]$ that equals $\pi r^2 - 1$ near $r = 0$. Note that if Z had negative self-intersection, one would have to take $f \leq 0$, thus making the integral of ρ over Z negative. Hence in this case $\omega_t + \kappa(t)\rho$ would cease to be symplectic for large $\kappa(t)$. □

Remark 2.1 (i) Once the family Z_t has been found we can alter J_t, keeping the Z_t J_t-holomorphic, so that ρ_t is semi-positive, i.e.

$$\rho_t(x, J_t x) \geq 0,$$

[4] If $A^2 = 0$ and A is not primitive, this may not hold because of problems in counting multiply-covered tori: see Taubes [24].

for all tangent vectors x. (One can achieve this by making both the fibers of the normal bundle and their orthogonal complements with respect to the closed form $d(r^2\gamma)$ J_t-invariant.) It follows that all the forms $\omega_t + \kappa(t)\rho_t$ tame J_t.

(ii) We will have to apply this process repeatedly, along families of submanifolds $Z_{1t}, \dots, Z_{nt}$ which intersect. The procedure as outlined above depends on the order chosen for the Z_{it}. However, with a little more care one can make the processes commute. (Alternatively, we can perform all inflations simultaneously.) Since we are dealing with a 1-parameter deformation, we may suppose that for each t the manifolds $Z_{jt}, j = 1, \dots, n$, meet transversally in pairs. If these intersections are all ω_t-orthogonal, then it is not hard to see that we can change the forms ρ_{it} so that

$$\omega_t + \sum_i \kappa_i(t)\rho_{it}$$

is symplectic for all $\kappa_i(t) \geq 0$. Indeed, to do this we just need to be careful near intersection points and here the local model is a product $U_i \times U_j$ with a product form, where $U_i \subset Z_{it}$. Therefore, if we choose the functions f_i above so that they vanish on U_i, the normal form τ for ω will respect this splitting, as will the forms ρ_{it}, ρ_{jt}. xs

To arrange that the intersections are orthogonal, one needs to perturb the families Z_{it}. Remark that two kinds of perturbation are needed here. By positivity of intersections, every intersection of Z_{it} with Z_{jt} counts positively and so, by a C^1-small perturbation we can arrange that these intersections are transverse for all i, j, t. (Observe that singularities can always be avoided for 1-parameter families, but possibly not for higher dimensional families.) We then need a large perturbation to make the intersections symplectically orthogonal: this can be done by the methods of [13]. Note that after these perturbations the resulting manifolds need no longer be J_t-holomorphic, though they will be holomorphic for some other family J_t'.

(iii) Another way of viewing this inflation process is as a form of the Gompf sum. To inflate along a connected submanifold Z one simply identifies X with a sum of the form[5]

$$X\#_{Z=Z_-}W,$$

where W is a ruled surface over Z which has a section Z_- with self-intersection equal to $-Z^2$, and then inflates X by increasing the size of the fiber of W. When Z has different components that intersect ω-orthogonally one needs to replace the Gompf sum by the kind of plumbing process used by McDuff-Symington in [18].

[5]This identification of X with $X\#_{Z=Z_-}W$ is a version of the "thinning" process of [18].

Gromov invariants of manifolds of nonsimple type

The following lemma contains all the information we need on Gromov invariants. We will denote the positive light cone by

$$\mathcal{P} = \{a \in H^2(X, \mathbf{R}) : a^2 > 0, a \cdot [\omega] > 0\},$$

and let $\overline{\mathcal{P}}$ be its closure. Recall that for any two nonzero elements $a, a' \in \overline{\mathcal{P}}$, we have $a \cdot a' \geq 0$ with equality only if $a = \lambda a'$ and $a^2 = 0$. (This is known as the light cone lemma.) Given $a \in H^2(X, \mathbf{Z})$, we will write $\mathrm{Gr}\,(a)$ instead of $\mathrm{Gr}\,(\mathrm{PD}(a))$.

Lemma 2.2 *Suppose that X is a symplectic manifold of nonsimple type. Then for every rational class $a \in \mathcal{P}$, $\mathrm{Gr}\,(qa) \neq 0$ for sufficiently large q. Moreover if*

$$a(E) \geq 0, \quad \textit{for all } E \in \mathcal{E},$$

the representation of $\mathrm{PD}(qa)$ by a J-holomorphic curve is connected and of multiplicity 1.

Proof: When $b_2^+ = 1$ the basic fact of Taubes–Seiberg–Witten theory is that for every class $a \in H^2(X, \mathbf{Z})$ there is a number $w(a)$ called the *wall-crossing* number of a such that

$$\mathrm{Gr}\,(a) \pm \mathrm{Gr}\,(K - a) = \pm w(a).$$

In particular, if $w(a) \neq 0$ and if $[\omega] \cdot (K - a) < 0$ then $\mathrm{Gr}\,(K - a)$ has to be zero, so that $\mathrm{Gr}\,(a) \neq 0$. When $b_1 = 0$, Kronheimer and Mrowka [4] showed that $w(a) = 1$ provided only that

$$k(a) = -K \cdot a + a^2 \geq 0.$$

Since $k(qa)$ is a quadratic function of q, the first statement holds in this case.

To prove the general case first recall that when $b_2^+ = 1$ the subspace of $H^2(X, \mathbf{R})$ generated by products of elements of H^1 has dimension at most 1. Moreover, when X is of nonsimple type and $H^1 \neq 0$ the dimension is exactly 1. Since $a^2 = 0$ for any $a \in (H^1)^2$, this subspace intersects $\overline{\mathcal{P}}$. Let a_0 be any integral element of $(H^1)^2 \cap \overline{\mathcal{P}}$. Then the wall-crossing formula of Li-Liu [7] says that for elements a with $k(a) \geq 0$, $w(a) \neq 0$ if and only if $a_0 \cdot (a - K/2) \neq 0$. Since $a_0 \cdot a \neq 0$ for $a \in \mathcal{P}$ by the light cone lemma, $w(qa) \neq 0$ for large q. The desired result follows readily.

To prove the second statement, observe that, because distinct classes in $\mathcal{P}$ have nontrivial intersection, the only way $\mathrm{PD}(a)$ can have a disconnected representative is if either $a^2 = 0$ or some of its components are exceptional divisors. But both of these possibilities are ruled out by our hypotheses. □

Proof of Theorem 1.9

If X has simple type then there only are a finite number of classes a with $\mathrm{Gr}_0(a) \neq 0$. Therefore V is finitely generated, by $a_1, \ldots, a_p$ say. It is then clear that there are smooth functions $c(t), \kappa_i(t) \geq 0$ such that

$$[\omega_0] = c(t)\left([\omega_t] + \sum_{i=1}^{p} \kappa_i(t) a_i\right).$$

Hence we can change ω_t to an isotopy by making p inflations along the classes $a_i, i = 1, \ldots, p$.

Proof of Theorem 1.2

Supose first that $[\omega_0]$ is rational. Whatever the intersection form Q_X, there is a basis of $H^2(X, \mathbf{Q})$ formed by rational classes $n[\omega_0], e_1, \ldots, e_k$ with $e_j^2 < 0$ for all j. Since $[\omega_0]^2 > 0$ we may choose the integer n so that the classes $n[\omega_0] \pm e_j \in \mathcal{P}$. Then, Lemma 2.2 implies that $\mathrm{Gr}\,(q(n[\omega_0] \pm e_j)) \neq 0$ for all j and large q. Now, given an isotopy ω_t with $[\omega_0] = [\omega_1]$, decompose its cohomology class as

$$[\omega_t] = c(t)[\omega_0] + \sum_j \lambda_j(t) e_j,$$

where $c(t) > 0$. By the openness of the set of symplectic forms, we can perturb ω_t so that the functions λ_j meet zero transversely. Taking $a = \mathrm{PD}(A)$ in Lemma 1.1 to be first $q(n[\omega_0] + e_1)$ and then $q(n[\omega_0] - e_1)$, we homotop ω_t so that $\lambda_1(t) \geq 0$ and then so that $\lambda_1(t) = 0$ for all t. (Since $[\omega_0](E) > 0$ for all $E \in \mathcal{E}$, Lemma 2.2(ii) tells us that we may ensure that the manifolds representing the duals of $q(n[\omega_0] \pm e_1)$ are connected by taking large enough n.) These homotopies will in general increase the function $c(t)$, but will not affect the λ_j for $j \neq 1$. Repeating this for $i = 2, \ldots, k$, we eventually homotop ω_t to a deformation such that $[\omega_t'] = c'(t)[\omega_0]$. Dividing by $c'(t)$ we find the desired isotopy.

We will deal with non rational $[\omega_0]$ by following a suggestion of Biran. Because $\mathcal{P}$ is open, it is easy to see that any class a in $\mathcal{P}$ is a positive sum of rational elements of $\mathcal{P}$, that is

$$a = \sum_{j=1}^{k+1} \lambda_j a_j, \quad \lambda_j \geq 0,$$

where each $a_j \in \mathcal{P} \cap H^2(X, \mathbf{Q})$. (Here $k = b_2^-$ as before.) Hence we may achieve the inflation along qa by performing $k+1$ inflations along suitable multiples of the a_j. $\square$

3 Embedding balls and blowing up

We sketch the steps in the proof of Corollary 1.5.

Step 1: Normalization. Choose an ω-compatible J on X which is integrable near k distinct point $x_1, \ldots, x_k$, and, for suitably small $\delta_i > 0$, fix a holomorphic and symplectic embedding

$$\iota : \coprod_{i=1}^{k} B(2\delta_i) \to X$$

which takes the center of the ith ball to x_i for all i. Given elements $g_j, j = 0, 1$ of $Emb(\coprod B(\lambda_i), X)$ we may, if the δ_i are small enough, isotop these embeddings so that they both extend ι.

Step 2: Forming the blow-up. We define the blow up $\widetilde{X}$ to be the manifold which is obtained by cutting out the balls $\iota(B(\delta_i))$ and identifying their boundaries to exceptional spheres Σ_i via the Hopf map. This carries a symplectic form that integrates on Σ_i to $\pi\delta_i^2$. (For more details of this step see [16, 15].) It remains to define symplectic forms on $\widetilde{X}$ corresponding to the g_j.

For some $\nu > 0$ choose an extension (also called g_j) of g_j to $\coprod B(\lambda_i + \nu)$. For $i = 1, \ldots, k$ let

$$\phi_i : B(\lambda_i + \nu) \to B(\lambda_i + \nu)$$

be a radial contraction which is the identity near the boundary and takes $B(\lambda_i)$ onto $B(\delta_i)$ by scalar multiplication. Define the map $\Phi_j : X \to X$ for $j = 0, 1$ by setting it equal to

$$\Phi_j = g_j \circ \phi_i \circ (g_j)^{-1} \quad \text{on} \quad g_j(B(\lambda_i + \nu)), 1 \leq i \leq k,$$

and extending by the identity. Put

$$\sigma_j = \Phi_j^*(\omega).$$

Then the forms σ_0 and σ_1 both equal the same multiple of the standard form near the balls $\iota(B(\delta_i))$ and so lift to forms which we will call $\widetilde{\sigma}_0, \widetilde{\sigma}_1$ on the blow-up $\widetilde{X}$. The manifold $(\widetilde{X}, \widetilde{\sigma}_j)$ is called the blow up of X by g_j. Note that the weight of the exceptional sphere Σ_i under $\widetilde{\sigma}_j$ is

$$\int_{\Sigma_i} \widetilde{\sigma}_j = \pi\lambda_i^2, \quad j = 0, 1.$$

Step 3: Isotopies in $\widetilde{X}$. Observe that g_j may be joined to

$$\iota' = \iota|_{\coprod B(\delta_i)}$$

by the family of embeddings

$$g_j^s : \coprod_{i=1}^{k} B(s\lambda_i) \to X.$$

Since the above blow-up construction can be done smoothly with respect to the parameter s, there is a deformation from $\widetilde{\sigma}_0$ to the blow-up of ι' and thence back to $\widetilde{\sigma}_1$. Therefore, by our main result, $\widetilde{\sigma}_0$ is isotopic to $\widetilde{\sigma}_1$ by some isotopy $\widetilde{\tau}_t$.

Let Σ be the union of the k exceptional spheres in $\widetilde{X}$. Both $\widetilde{\sigma}_0$ and $\widetilde{\sigma}_1$ are equal and nondegenerate near Σ by construction. Moreover, we can arrange that the $\widetilde{\tau}_t$ are also nondegenerate on Σ. To see this, observe that the deformation between $\widetilde{\sigma}_0$ and $\widetilde{\sigma}_1$ that was described above consists of forms that are nondegenerate on Σ. Therefore, when constructing the inflation we can choose the family J_t so that each component of Σ is J_t-holomorphic. (It is still possible to get sufficiently generic J_t's satisfying this restriction.) Then, by using the techniques mentioned in Remark 2.1 it is easy to arrange that the $\widetilde{\tau}_t$ are nondegenerate on Σ. Therefore, there is a family of diffeomorphisms $\tilde{h}_t : (X, \Sigma) \to (X, \Sigma)$ such that $\tilde{h}_t^*(\widetilde{\tau}_t) = \widetilde{\sigma}_0$ on some neighborhood of $\mathcal{N}_\varepsilon$ of Σ. We may also assume that $\tilde{h}_0 = \mathrm{id}$ and that $\tilde{h}_1 = \mathrm{id}$ on $\mathcal{N}_\varepsilon$. Hence there is a family $\tilde{\psi}_t$ of diffeomorphisms of $\widetilde{X}$ such that

$$\tilde{\psi}_0 = \mathrm{id}, \quad \tilde{\psi}_t = \mathrm{id} \ \text{ on } \ \mathcal{N}_\varepsilon, \quad \tilde{\psi}_1^* \tilde{h}_1^*(\widetilde{\sigma}_1) = \widetilde{\sigma}_0.$$

Step 4. Because the forms $\widetilde{\tau}_t$ are constant near $\mathcal{N}_\varepsilon$, they are the blow-ups of corresponding forms τ_t on X. Similarly, because the diffeomorphisms $\tilde{\psi}_t$ and $\tilde{h}_1$ are the identity near $\mathcal{N}_\varepsilon$, they are the blow-up of corresponding diffeomorphisms ψ_t, h_1 on X which are the identity near the balls $\iota(B(\delta_i))$. Therefore, by construction of σ_0, σ_1 we find that

$$\sigma_0 = \Phi_0^*(\omega) = \psi_1^* h_1^* \Phi_1^*(\omega) = \psi_1^* h_1^* \sigma_1.$$

Moreover it is easy to check that

$$F = \Phi_1 \circ h_1 \circ \psi_1 \circ \Phi_0^{-1}$$

is a symplectomorphism of X such that $F \circ g_0 = g_1$.

Hence it remains to show that F is isotopic to the identity. But such an isotopy F^s can be constructed by doing the above construction for each s where $g_j^s, s \le 1$, is as above. For s small enough one finds that $g_0^s = g_1^s$ which means that F^s is the identity. There is one point worthy of note in this last step. In order to construct F we inflated a single deformation into an isotopy, using 1-parameter families of submanifolds. In fact, as pointed out in Remark 2.1(ii)

we just need to do one inflation process along a family of orthogonally intersecting submanifolds Z_t. In order to inflate an arbitrary 1-parameter family of deformations one would have to use a 2-parameter family Z_{st} of submanifolds, and, in general, these might encounter singularities. However, in the present situation the family of deformations is not at all arbitrary, but consists in lopping off the ends of the original deformation until one arrives just at the center point. Therefore, as is not hard to check, the corresponding family of inflations can be constructed using the original 1-parameter family Z_t. □

4 Symplectic fibrations

Let ω_0, ω_1 be π-compatible forms on the oriented fibration $\pi : X \to B$. Our aim is to find conditions under which these forms are either isotopic or deformation equivalent. Throughout this section we assume that B has dimension 2. At each point $p \in X$ we define H_p to be the ω_0-horizontal space, ie the ω_0-orthogonal complement to the tangent space to the fiber through p. This space has a natural orientation.

Lemma 4.1 *In the above situation, assume that the restriction of ω_1 to H_p is a nondegenerate and positive form for all $p \in X$. Suppose further that either F has dimension 2 or that the restrictions to each fiber of ω_0 and ω_1 are equal. Then the forms*

$$\omega_t = (1-t)\omega_0 + t\omega_1, \quad 0 \le t \le 1,$$

are all nondegenerate. Hence if in addition ω_0, ω_1 are cohomologous, they are isotopic.

Proof: This follows from a simple calculation done at each point p. Let us suppose that ω_0 and ω_1 restrict to the form ρ on the fiber F_p through p. Then we may choose coordinates near p so that

$$\omega_0 = \rho + dx \wedge dy, \qquad \omega_1 = \rho + a dx \wedge dy + dx \wedge \alpha - dy \wedge \beta,$$

where α, β are 1-forms on F_p. By hypothesis, $a > 0$. It is easy to check that

$$\omega_1^n = n(a+\kappa)\rho^{n-1} \wedge dx \wedge dy$$

where the function κ is defined by $(n-1)\rho^{n-2}\alpha \wedge \beta = \kappa\rho^{n-1}$. Thus we must have $a + \kappa > 0$. Further,

$$\omega_t^n = n(1 - t + at + t^2\kappa)\rho^{n-1} \wedge dx \wedge dy,$$

which is always nondegenerate because $|\kappa|t^2 < at \le 1 - t + at$ when $0 \le t \le 1$. The calculation when F has dimension 2 is even easier. □

Remark 4.2 The above argument does not always go through when $a < 0$. For example, $1 - t + at + t^2\kappa < 0$ if $a = -7, \kappa = 8, t = 1/2$. So the set of cohomologous fibered forms need not be convex.

Corollary 4.3 *Given any π-compatible form ω, and any nondegenerate 2-form σ on the base B, the forms $\omega_\mu = \omega + \mu\pi^*\sigma$ are nondegenerate for all $\mu > 0$. Moreover if H_p is defined with respect to any π-compatible form ω_0 the restriction of ω_μ to H_p is nondegenerate and positive when μ is sufficiently large.*

Proof: This is immediate since in the previous lemma we may identify $dx \wedge dy$ with $\pi^*(\sigma)$. □

The above results immediately imply:

Proposition 4.4 *Let $\pi : X \to B$ be an oriented fibration with base B of dimension 2, and let ω_0 and ω_1 be any two π-compatible forms on X. If either F has dimension 2 or ω_0 and ω_1 have equal restrictions to all fibers, the forms ω_0 and ω_1 are deformation equivalent through π-compatible forms.*

Proof of Proposition 1.10

We are given a π-compatible form ω om $X = F \times B$ in the class of $\sigma_F + \lambda\sigma_B$. By Proposition 4.4 ω is deformation equivalent to the split form $\sigma_F + \lambda\sigma_B$. Thus we just have to convert this deformation into an isotopy. We may assume that B is not a sphere since that case is already known. If B is also not a torus, X has the smooth structure of a minimal Kähler surface of general type, and so, by [22] its only non-zero Gromov invariant is that of the canonical class K. Thus V is generated by the class

$$\mathrm{PD}(K) = -c_1(TM, J) = (2g_F - 2)[\sigma_F] + (2g_B - 2)[\sigma_B].$$

Since our assumptions imply that $[\omega] \in \mathbf{R}^+[\omega_t] + \mathbf{R}^+\mathrm{PD}(K)$, the result follows from Theorem 1.9. It remains to consider the case when B is a torus. Taubes showed in [24] that $\mathrm{Gr}\,(B) = 2 - 2g_F$, so that if $g_F > 1$, V is generated by $[\sigma_F]$. Hence the result follows as before. □

The above argument applies to more general fibrations. However, it gives no information about the manifold T^4, and also does not deal with the question as to which symplectic forms are isotopic to π-compatible forms.

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